

Reconstruction of a Low-rank Matrix in the Presence of Gaussian Noise

Andrey Shabalin and Andrew Nobel

July 26, 2010

Abstract

In this paper we study the problem of reconstruction of a low-rank matrix observed with additive Gaussian noise. First we show that under mild assumptions (about the prior distribution of the signal matrix) we can restrict our attention to reconstruction methods that are based on the singular value decomposition of the observed matrix and act only on its singular values (preserving the singular vectors). Then we determine the effect of noise on the SVD of low-rank matrices by building a connection between matrix reconstruction problem and spiked population model in random matrix theory. Based on this knowledge, we propose a new reconstruction method, called RMT, that is designed to reverse the effect of the noise on the singular values of the signal matrix and adjust for its effect on the singular vectors. With an extensive simulation study we show that the proposed method outperform even oracle versions of both soft and hard thresholding methods and closely matches the performance of a general oracle scheme.

1 Introduction

Existing and emerging technologies provide scientists with access to a growing wealth of data. Some data is initially produced in the matrix form, while other can be represented in a matrix form once the data from multiple samples is combined. The data is often measured with noise due to limitations of the data generating technologies. To reduce the noise we need some information about the possible structure of signal component. In this paper

we assume the signal to be a low rank matrix. Assumption of this sort appears in multiple fields of study including genomics (Raychaudhuri et al., 2000; Alter et al., 2000; Holter et al., 2000; Wall et al., 2001; Troyanskaya et al., 2001), compressed sensing (Candès et al., 2006; Candès and Recht, 2009; Candès and Recht; Donoho, 2006), and image denoising (Wongsawat, Rao, and Oraintara; Konstantinides et al., 1997). In many cases the signal matrix is known to have low rank. For example, a matrix of squared distances between points in d -dimensional Euclidean space is known to have rank at most $d + 2$. A correlation matrix for a set of points in d -dimensional Euclidean space has rank at most d . In other cases the target matrix is often assumed to have low rank, or to have a good low-rank approximation.

In this paper we address the problem of recovering a low rank signal matrix whose entries are observed in the presence of additive Gaussian noise. The reconstruction problem considered here has a signal plus noise structure. Our goal is to recover an unknown $m \times n$ matrix A of low rank that is observed in the presence of i.i.d. Gaussian noise as matrix Y :

$$Y = A + \frac{\sigma}{\sqrt{n}}W, \quad \text{where } W_{ij} \sim \text{i.i.d. } N(0, 1).$$

The factor $n^{-1/2}$ ensures that the signal and noise are comparable, and is employed for the asymptotic study of matrix reconstruction in Section 3. In what follows, we first consider the variance of the noise σ^2 to be known, and assume that it is equal to one. (In Section 4.1 we propose an estimator for σ , which we use in the proposed reconstruction method.) In this case the model (1) simplifies to

$$Y = A + \frac{1}{\sqrt{n}}W, \quad \text{where } W_{ij} \sim \text{i.i.d. } N(0, 1). \quad (1)$$

Formally, a matrix recovery scheme is a map $g: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ from the space of $m \times n$ matrices to itself. Given a recovery scheme $g(\cdot)$ and an observed matrix Y from the model (1), we regard $\widehat{A} = g(Y)$ as an estimate of A , and measure the performance of the estimate \widehat{A} by

$$\text{Loss}(A, \widehat{A}) = \|\widehat{A} - A\|_F^2, \quad (2)$$

where $\|\cdot\|_F$ denotes the Frobenius norm. The Frobenius norm of an $m \times n$ matrix $B = \{b_{ij}\}$ is given by

$$\|B\|_F^2 = \sum_{i=1}^m \sum_{j=1}^n b_{ij}^2.$$

Note that if the vector space $\mathbb{R}^{m \times n}$ is equipped with the inner product $\langle A, B \rangle = \text{tr}(A'B)$, then $\|B\|_F^2 = \langle B, B \rangle$.

1.1 Hard and Soft Thresholding

A natural starting point for reconstruction of the target matrix A in (1) is the singular value decomposition (SVD) of the observed matrix Y . Recall that the singular value decomposition of an $m \times n$ matrix Y is given by the factorization

$$Y = UDV' = \sum_{j=1}^{m \wedge n} d_j u_j v'_j.$$

Here U is an $m \times m$ orthogonal matrix whose columns are the left singular vectors u_j , V is an $n \times n$ orthogonal matrix whose columns are the right singular vectors v_j , and D is an $m \times n$ matrix with singular values $d_j = D_{jj} \geq 0$ on the diagonal and all other entries equal to zero. Although it is not necessarily square, we will refer to D as a diagonal matrix and write $D = \text{diag}(d_1, \dots, d_{m \wedge n})$, where $m \wedge n$ denotes the minimum of m and n .

Many matrix reconstruction schemes act by shrinking the singular values of the observed matrix towards zero. Shrinkage is typically accomplished by hard or soft thresholding. Hard thresholding schemes set every singular value of Y less than a given positive threshold λ equal to zero, leaving other singular values unchanged. The family of hard thresholding schemes is defined by

$$g_\lambda^H(Y) = \sum_{j=1}^{m \wedge n} d_j I(d_j \geq \lambda) u_j v'_j, \quad \lambda > 0.$$

Soft thresholding schemes subtract a given positive number ν from each singular value, setting values less than ν equal to zero. The family of soft thresholding schemes is defined by

$$g_\nu^S(Y) = \sum_{j=1}^{m \wedge n} (d_j - \nu)_+ u_j v'_j, \quad \nu > 0.$$

Hard and soft thresholding schemes can be defined equivalently in the penalized forms

$$g_\lambda^H(Y) = \arg \min_B \{ \|Y - B\|_F^2 + \lambda^2 \text{rank}(B) \}$$

$$g_\nu^S(Y) = \arg \min_B \{ \|Y - B\|_F^2 + 2\nu \|B\|_*\}.$$

In the second display, $\|B\|_*$ denotes the nuclear norm of B , which is equal to the sum of its singular values.

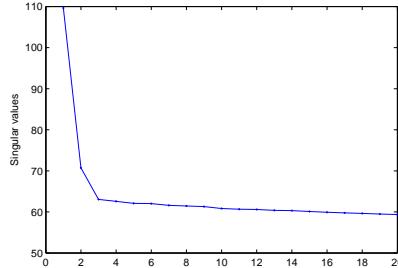


Figure 1: Scree plot for a 1000×1000 rank 2 signal matrix with noise.

In practice, hard and soft thresholding schemes require estimates of the noise variance, as well as the selection of appropriate cutoff or shrinkage parameters. There are numerous methods in the literature for choosing the hard threshold λ . Heuristic methods often make use of the scree plot, which displays the singular values of Y in decreasing order: λ is typically chosen to be the y-coordinate of a well defined “elbow” in the resulting curve. A typical scree plot for a 1000×1000 matrix with rank 2 signal is shown in Figure 1. The “elbow” point of the curve on the plot clearly indicate that the signal has rank 2.

A theoretically justified selection of hard threshold λ is presented in recent work of [Bunea et al. \(2010\)](#). They also provide performance guarantees for the resulting hard thresholding scheme using techniques from empirical process theory and complexity regularization. Selection of the soft thresholding shrinkage parameter ν may also be accomplished by a variety of methods. [Negahban and Wainwright \(2009\)](#) propose a specific choice of ν and provide performance guarantees for the resulting soft thresholding scheme.

Figure 2 illustrates the action of hard and soft thresholding on a 1000×1000 matrix with a rank 50 signal. The blue line indicates the singular values of the signal A and the green line indicates the those of the observed matrix Y . The plots show the singular values of the hard and soft thresholding estimates incorporating the best choice of the parameters λ and ν , respectively. It is evident from the figure that neither thresholding scheme delivers an accurate

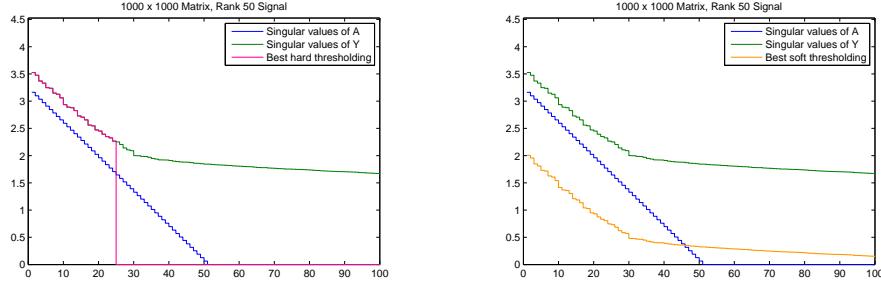


Figure 2: Singular values of hard and soft thresholding estimates.

estimate of the signal's singular values. Moreover examination of the loss indicates that they do not provide a good estimates of the signal matrix.

The families of hard and soft thresholding methods encompass many existing reconstruction schemes. Both thresholding approaches seek low rank (sparse) estimates of the target matrix, and both can be naturally formulated as optimization problems. However, the family of all reconstruction schemes is much larger, and it is natural to consider alternatives to hard and soft thresholding that may offer better performance.

In this paper, we start with a principled analysis of the matrix reconstruction problem, with the effort of making as few assumptions as possible. Theoretically motivated design of the method. Based on the analysis of the reconstruction problem and, in particular, the analysis of the effect of noise on low-rank matrices we design a new reconstruction method with a theoretically motivated design.

1.2 Outline

We start the paper with an analysis of the finite sample properties of the matrix reconstruction problem. The analysis does not require the matrix A to have low rank and only requires that the distribution of noise matrix W is orthogonally invariant (does not change under left and right multiplications by orthogonal matrices). Under mild conditions (the prior distribution of A must be orthogonally invariant) we prove that we can restrict our attention to the reconstruction methods that are based on the SVD of the observed matrix Y and act only on its singular values, not affecting the singular vectors.

This result has several useful consequences. First, it reduces the space of reconstruction schemes we consider from $g : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ to just $\mathbb{R}^{m \wedge n} \rightarrow \mathbb{R}^{m \wedge n}$. Moreover, it gives us the recipe for design of the new reconstruction scheme: we determine the effect of the noise on the singular values and the singular value decomposition of the signal matrix A and then built the new reconstruction method to reverse the effect of the noise on the singular values of A and account for its effect on the singular vectors.

To determine the effect of noise on low-rank signal we build a connection between the matrix reconstruction problem and spiked population models in random matrix theory. Spiked population models were introduced by [Johnstone \(2001\)](#). The asymptotic matrix reconstruction model that matches the setup of spiked population models assumes the rank of A and its non-zero singular values to be fixed as the matrix dimensions to grow at the same rate: $m, n \rightarrow \infty$ and $m/n \rightarrow c > 0$. We use results from random matrix theory about the limiting behavior of the eigenvalues ([Marčenko and Pastur, 1967](#); [Wachter, 1978](#); [Geman, 1980](#); [Baik and Silverstein, 2006](#)) and eigenvectors ([Paul, 2007](#); [Nadler, 2008](#); [Lee et al., 2010](#)) of sample covariance matrices in spiked population models to determine the limiting behavior of the singular values and the singular vectors of matrix Y .

We apply these results to design a new matrix reconstruction method, which we call RMT for its use of random matrix theory. The method estimated the singular values of A from the singular values of Y and applies additional shrinkage to them to correct for the difference between the singular vectors of Y and those of A . The method uses an estimator of the noise variance which is based on the sample distribution of the singular values of Y corresponding to the zero singular values of A .

We conduct an extensive simulation study to compare RMT method against the oracle versions of hard and soft thresholding and the orthogonally invariant oracle. We run all four methods on matrices of different sizes, with signals of various ranks and spectra. The simulations clearly show that RMT method strongly outperforms the oracle versions of both hard and soft thresholding methods and closely matches the performance of the orthogonally invariant oracle (the oracle scheme that acts only on the singular values of the observed matrix).

The paper is organized as follows. In Section 2 we present the analysis of the finite sample properties of the reconstruction problem. In Section 3 we determine the effect of the noise on the singular value decomposition of

row rank matrices. In Section 4 we construct the proposed reconstruction method based on the results of the previous section. The method employs the noise variance estimator presented in Section 4.1. Finally, in Section 5 we present the simulation study comparing RMT method to the oracle versions of hard and soft thresholding and the orthogonally invariant oracle method.

2 Orthogonally Invariant Reconstruction Methods

The additive model (1) and Frobenius loss (2) have several elementary invariance properties, that lead naturally to the consideration of reconstruction methods with analogous forms of invariance. Recall that a square matrix U is said to be orthogonal if $UU' = U'U = I$, or equivalently, if the rows (or columns) of U are orthonormal. If we multiply each side of (1) from the left right by orthogonal matrices U and V' of appropriate dimensions, we obtain

$$UYV' = UAV' + \frac{1}{\sqrt{n}}UWV'. \quad (3)$$

Proposition 1. *Equation (3) is a reconstruction problem of the form (1) with signal UAV' and observed matrix UYV' . If \widehat{A} is an estimate of A in model (1), then $U\widehat{A}V'$ is an estimate of UAV' in model (3) with the same loss.*

Proof. If A has rank r then UAV' also has rank r . Thus prove the first statement, it suffices to show that UWV' in (3) has independent $N(0, 1)$ entries. This follows from standard properties of the multivariate normal distribution. In order to establish the second statement of the proposition, let U and V be the orthogonal matrices in (3). For any $m \times n$ matrix B ,

$$\|UB\|_F^2 = \text{tr}[(UB)'(UB)] = \text{tr}[B'B] = \|B\|_F^2,$$

and more generally $\|UBV'\|_F^2 = \|B\|_F^2$. Applying the last equality to $B = \widehat{A} - A$ yields

$$\text{Loss}(UAV', U\widehat{A}V') = \|U(\widehat{A} - A)V'\|_F^2 = \|\widehat{A} - A\|_F^2 = \text{Loss}(A, \widehat{A})$$

as desired. \square

In the proof we use the fact that the distribution of matrix W does not change under left and right multiplications by orthogonal matrices. We will call such distributions orthogonally invariant.

Definition 2. *A random $m \times n$ matrix Z has an orthogonally invariant distribution if for any orthogonal matrices U and V of appropriate size the distribution of UZV' is the same as the distribution of Z .*

In light of Proposition 1 it is natural to consider reconstruction schemes those action do not change under orthogonal transformations of the reconstruction problem.

Definition 3. *A reconstruction scheme $g(\cdot)$ is orthogonally invariant if for any $m \times n$ matrix Y , and any orthogonal matrices U and V of appropriate size, $g(UYV') = Ug(Y)V'$.*

In general, a good reconstruction method need not be orthogonally invariant. For example, if the signal matrix A is known to be diagonal, then for each Y the estimate $g(Y)$ should be diagonal as well, and in this case $g(\cdot)$ is not orthogonally invariant. However, as we show in the next theorem, if we have no information about the singular vectors of A (either prior information or information from the singular values of A), then it suffices to restrict our attention to orthogonally invariant reconstruction schemes.

Theorem 4. *Let $Y = \mathbf{A} + W$, where \mathbf{A} is a random target matrix. Assume that \mathbf{A} and W are independent and have orthogonally invariant distributions. Then, for every reconstruction scheme $g(\cdot)$, there is an orthogonally invariant reconstruction scheme $\tilde{g}(\cdot)$ whose expected loss is the same, or smaller, than that of $g(\cdot)$.*

Proof. Let \mathbf{U} be an $m \times m$ random matrix that is independent of \mathbf{A} and W , and is distributed according to Haar measure on the compact group of $m \times m$ orthogonal matrices. Haar measure is (uniquely) defined by the requirement that, for every $m \times m$ orthogonal matrix C , both $C\mathbf{U}$ and $\mathbf{U}C$ have the same distribution as \mathbf{U} (c.f. [Hofmann and Morris, 2006](#)). Let \mathbf{V} be an $n \times n$ random matrix distributed according to the Haar measure on the compact group of $n \times n$ orthogonal matrices that is independent of \mathbf{A} , W and \mathbf{U} . Given a reconstruction scheme $g(\cdot)$, define a new scheme

$$\tilde{g}(Y) = \mathbb{E}[\mathbf{U}'g(\mathbf{U}Y\mathbf{V}')\mathbf{V} \mid Y].$$

It follows from the definition of \mathbf{U} and \mathbf{V} that $\tilde{g}(\cdot)$ is orthogonally invariant. The independence of $\{\mathbf{U}, \mathbf{V}\}$ and $\{\mathbf{A}, W\}$ ensures that conditioning on Y is equivalent to conditioning on $\{\mathbf{A}, W\}$, which yields the equivalent representation

$$\tilde{g}(Y) = \mathbb{E}[\mathbf{U}'g(\mathbf{U}Y\mathbf{V}')\mathbf{V} \mid \mathbf{A}, W].$$

Therefore,

$$\begin{aligned}\mathbb{E} \text{Loss}(\mathbf{A}, \tilde{g}(Y)) &= \mathbb{E} \left\| \mathbb{E}[\mathbf{U}'g(\mathbf{U}Y\mathbf{V}')\mathbf{V} - \mathbf{A} \mid \mathbf{A}, W] \right\|_F^2 \\ &\leq \mathbb{E} \left\| \mathbf{U}'g(\mathbf{U}Y\mathbf{V}')\mathbf{V} - \mathbf{A} \right\|_F^2 \\ &= \mathbb{E} \left\| g(\mathbf{U}Y\mathbf{V}') - \mathbf{U}\mathbf{A}\mathbf{V}' \right\|_F^2,\end{aligned}$$

the inequality follows from the conditional version of Jensen's inequality applied to each term in the sum defining the squared norm. The final equality follows from the orthogonality of \mathbf{U} and \mathbf{V} . The last term in the previous display can be analyzed as follows:

$$\begin{aligned}\mathbb{E} \left\| g(\mathbf{U}Y\mathbf{V}') - \mathbf{U}\mathbf{A}\mathbf{V}' \right\|_F^2 &= \mathbb{E} \left[\mathbb{E} \left(\left\| g(\mathbf{U}\mathbf{A}\mathbf{V}' + n^{-1/2}\mathbf{U}W\mathbf{V}') - \mathbf{U}\mathbf{A}\mathbf{V}' \right\|_F^2 \mid \mathbf{U}, \mathbf{V}, \mathbf{A} \right) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left(\left\| g(\mathbf{U}\mathbf{A}\mathbf{V}' + n^{-1/2}W) - \mathbf{U}\mathbf{A}\mathbf{V}' \right\|_F^2 \mid \mathbf{U}, \mathbf{V}, \mathbf{A} \right) \right] \\ &= \mathbb{E} \left\| g(\mathbf{U}\mathbf{A}\mathbf{V}' + n^{-1/2}W) - \mathbf{U}\mathbf{A}\mathbf{V}' \right\|_F^2.\end{aligned}$$

The first equality follows from the definition of Y ; the second follows from the independence of W and $\mathbf{U}, \mathbf{A}, \mathbf{V}$, and the orthogonal invariance of $\mathcal{L}(W)$. By a similar argument using the orthogonal invariance of $\mathcal{L}(\mathbf{A})$, we have

$$\begin{aligned}\mathbb{E} \left\| g(\mathbf{U}\mathbf{A}\mathbf{V}' + n^{-1/2}W) - \mathbf{U}\mathbf{A}\mathbf{V}' \right\|_F^2 &= \mathbb{E} \left[\mathbb{E} \left(\left\| g(\mathbf{U}\mathbf{A}\mathbf{V}' + n^{-1/2}W) - \mathbf{U}\mathbf{A}\mathbf{V}' \right\|_F^2 \mid \mathbf{U}, \mathbf{V}, W \right) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left(\left\| g(\mathbf{A} + n^{-1/2}W) - \mathbf{A} \right\|_F^2 \mid \mathbf{U}, \mathbf{V}, W \right) \right] \\ &= \mathbb{E} \left\| g(\mathbf{A} + n^{-1/2}W) - \mathbf{A} \right\|_F^2.\end{aligned}$$

The final term above is $\mathbb{E} \text{Loss}(\mathbf{A}, g(Y))$. This completes the proof. \square

Based on Theorem 4 will restrict our attention to orthogonally invariant reconstruction schemes in what follows.

As noted in introduction, the singular value decomposition (SVD) of the observed matrix Y is a natural starting point for reconstruction of a signal matrix A . As we show below, the SVD of Y is intimately connected with orthogonally invariant reconstruction methods. An immediate consequence of the decomposition $Y = UDV'$ is that $U'YV = D$, so we can diagonalize Y by means of left and right orthogonal multiplications.

The next proposition follows from our ability to diagonalize the signal matrix A in the reconstruction problem.

Proposition 5. *Let $Y = A + n^{-1/2}W$, where W has an orthogonally invariant distribution. If $g(\cdot)$ is an orthogonally invariant reconstruction scheme, then for any fixed signal matrix A , the distribution of $\text{Loss}(A, g(Y))$, and in particular $\mathbb{E}\text{Loss}(A, g(Y))$, depends only on the singular values of A .*

Proof. Let $UDAV'$ be the SVD of A . Then $D_A = U'AV$, and as the Frobenius norm is invariant under left and right orthogonal multiplications,

$$\begin{aligned}\text{Loss}(A, g(Y)) &= \|g(Y) - A\|_F^2 = \|U'(g(Y) - A)V\|_F^2 \\ &= \|U'g(Y)V - U'AV\|_F^2 = \|g(U'YV) - D_A\|_F^2 \\ &= \|g(D_A + n^{-1/2}U'WV) - D_A\|_F^2.\end{aligned}$$

The result now follows from the fact that UWV' has the same distribution as W . \square

We now address the implications of our ability to diagonalize the observed matrix Y . Let $g(\cdot)$ be an orthogonally invariant reconstruction method, and let UDV' be the singular value decomposition of Y . It follows from the orthogonal invariance of $g(\cdot)$ that

$$g(Y) = g(UDV') = Ug(D)V' = \sum_{i=1}^m \sum_{j=1}^n c_{ij} u_i v'_j \quad (4)$$

where c_{ij} depend only on the singular values of Y . In particular, any orthogonally invariant $g(\cdot)$ reconstruction method is completely determined by how it acts on diagonal matrices. The following theorem allows us to substantially refine the representation (4).

Theorem 6. *Let $g(\cdot)$ be an orthogonally invariant reconstruction scheme. Then $g(Y)$ is diagonal whenever Y is diagonal.*

Proof. Assume without loss of generality that $m \geq n$. Let the observed matrix $Y = \text{diag}(d_1, d_2, \dots, d_n)$, and let $\widehat{A} = g(Y)$ be the reconstructed matrix. Fix a row index $1 \leq k \leq m$. We will show that $\widehat{A}_{kj} = 0$ for all $j \neq k$. Let D_L be an $m \times m$ matrix derived from the identity matrix by flipping the sign of the k^{th} diagonal element. More formally, $D_L = I - 2e_k e_k'$, where e_k is the k^{th} standard basis vector in \mathbb{R}^m . The matrix D_L is known as a Householder reflection.

Let D_R be the top left $n \times n$ submatrix of D_L . Clearly $D_L D_L' = I$ and $D_R D_R' = I$, so both D_L and D_R are orthogonal. Moreover, all three matrices D_L, Y , and D_R are diagonal, and therefore we have the identity $Y = D_L Y D_R$. It then follows from the orthogonal invariance of $g(\cdot)$ that

$$\widehat{A} = g(Y) = g(D_L Y D_R) = D_L g(Y) D_R = D_L \widehat{A} D_R.$$

The $(i, j)^{\text{th}}$ element of the matrix $D_L \widehat{A} D_R$ is $\widehat{A}_{ij} (-1)^{\delta_{ik}} (-1)^{\delta_{jk}}$, and therefore $\widehat{A}_{kj} = -\widehat{A}_{kj}$ if $j \neq k$. As k was arbitrary, \widehat{A} is diagonal. \square

As an immediate corollary of Theorem 6 and equation (4) we obtain a compact, and useful, representation of any orthogonally invariant reconstruction scheme $g(\cdot)$.

Corollary 7. *Let $g(\cdot)$ be an orthogonally invariant reconstruction method. If the observed matrix Y has singular value decomposition $Y = \sum d_j u_j v_j'$ then the reconstructed matrix has the form*

$$\widehat{A} = g(Y) = \sum_{j=1}^{m \wedge n} c_j u_j v_j', \quad (5)$$

where the coefficients c_j depend only on the singular values of Y .

The converse of Corollary 7 is true under a mild additional condition. Let $g(\cdot)$ be a reconstruction scheme such that $g(Y) = \sum c_j u_j v_j'$, where $c_j = c_j(d_1, \dots, d_{m \wedge n})$ are fixed functions of the singular values of Y . If the functions $\{c_j(\cdot)\}$ are such that $c_i(d) = c_j(d)$ whenever $d_i = d_j$, then $g(\cdot)$ is orthogonally invariant. This follows from the uniqueness of the singular value decomposition.

3 Asymptotic Matrix Reconstruction and Random Matrix Theory

Random matrix theory is broadly concerned with the spectral properties of random matrices, and is an obvious starting point for an analysis of matrix reconstruction. The matrix reconstruction problem has several points of intersection with random matrix theory. Recently a number of authors have studied low rank deformations of Wigner matrices (Capitaine et al., 2009; Féral and Péché, 2007; Maida, 2007; Péché, 2006). However, their results concern symmetric matrices, a constraint not present in the reconstruction model, and are not directly applicable to the reconstruction problem of interest here. (Indeed, our simulations of non-symmetric matrices exhibit behavior deviating from that predicted by the results of these papers.) A signal plus noise framework similar to matrix reconstruction is studied in Dozier and Silverstein (2007) and Nadakuditi and Silverstein (2007), however both these papers model the signal matrix to be random, while in the matrix reconstruction problem we assume it to be non-random. El Karoui (2008) considered the problem of estimation the eigenvalues of a population covariance matrix from a sample covariance matrix, which is similar to the problem of estimation of the singular values of A from the singular values of Y . However for the matrix reconstruction problem it is equally important to estimate the difference between the singular vectors of A and Y , in addition to the estimate of the singular values of A .

Our proposed denoising scheme is based on the theory of spiked population models in random matrix theory. Using recent results on spiked population models, we establish asymptotic connections between the singular values and vectors of the signal matrix A and those of the observed matrix Y . These asymptotic connections provide us with finite-sample estimates that can be applied in a non-asymptotic setting to matrices of small or moderate dimensions.

3.1 Asymptotic Matrix Reconstruction Model

The proposed reconstruction method is derived from an asymptotic version of the matrix reconstruction problem (1). For $n \geq 1$ let integers $m = m(n)$ be defined in such a way that

$$\frac{m}{n} \rightarrow c > 0 \text{ as } n \rightarrow \infty. \quad (6)$$

For each n let Y , A , and W be $m \times n$ matrices such that

$$Y = A + \frac{1}{\sqrt{n}}W, \quad (7)$$

where the entries of W are independent $N(0, 1)$ random variables. We assume that the signal matrix A has fixed rank $r \geq 0$ and fixed non-zero singular values $\lambda_1(A), \dots, \lambda_r(A)$ that are independent of n . The constant c represents the limiting aspect ratio of the observed matrices Y . The scale factor $n^{-1/2}$ ensures that the singular values of the signal matrix are comparable to those of the noise. We note that Model (7) matches the asymptotic model used by [Capitaine et al. \(2009\)](#); [Féral and Péché \(2007\)](#) in their study of fixed rank perturbations of Wigner matrices.

In what follows $\lambda_j(B)$ will denote the j -th singular value of a matrix B , and $u_j(B)$ and $v_j(B)$ will denote, respectively, the left and right singular values corresponding to $\lambda_j(B)$. Our first proposition concerns the behavior of the singular values of Y when the signal matrix A is equal to zero.

Proposition 8. *Under the asymptotic reconstruction model with $A = 0$ the empirical distribution of the singular values $\lambda_1(Y) \geq \dots \geq \lambda_{m \wedge n}(Y)$ converges weakly to a (non-random) limiting distribution with density*

$$f_Y(s) = \frac{s^{-1}}{\pi(c \wedge 1)} \sqrt{(a - s^2)(s^2 - b)}, \quad s \in [\sqrt{a}, \sqrt{b}], \quad (8)$$

where $a = (1 - \sqrt{c})^2$ and $b = (1 + \sqrt{c})^2$. Moreover, $\lambda_1(Y) \xrightarrow{P} 1 + \sqrt{c}$ and $\lambda_{m \wedge n}(Y) \xrightarrow{P} 1 - \sqrt{c}$ as n tends to infinity.

The existence and form of the density $f_Y(\cdot)$ are a consequence of the classical Marčenko-Pastur theorem ([Marčenko and Pastur, 1967](#); [Wachter, 1978](#)). The in-probability limits of $\lambda_1(Y)$ and $\lambda_{m \wedge n}(Y)$ follow from later work of [Geman \(1980\)](#) and [Wachter \(1978\)](#), respectively. If $c = 1$, the density function $f_Y(s)$ simplifies to the quarter-circle law $f_Y(s) = \pi^{-1} \sqrt{4 - s^2}$ for $s \in [0, 2]$.

The next two results concern the limiting eigenvalues and eigenvectors of Y when A is non-zero. Proposition 9 relates the limiting eigenvalues of Y to the (fixed) eigenvalues of A , while Proposition 10 relates the limiting singular vectors of Y to the singular vectors of A . Proposition 9 is based on recent work of [Baik and Silverstein \(2006\)](#), while Proposition 10 is based on recent work of [Paul \(2007\)](#), [Nadler \(2008\)](#), and [Lee et al. \(2010\)](#). The proofs of both results are given in Section 5.5.

Proposition 9. Let Y follow the asymptotic matrix reconstruction model (7) with signal singular values $\lambda_1(A) \geq \dots \geq \lambda_r(A) > 0$. For $1 \leq j \leq r$, as n tends to infinity,

$$\lambda_j(Y) \xrightarrow{P} \begin{cases} \left(1 + \lambda_j^2(A) + c + \frac{c}{\lambda_j^2(A)}\right)^{1/2} & \text{if } \lambda_j(A) > \sqrt[4]{c} \\ 1 + \sqrt{c} & \text{if } 0 < \lambda_j(A) \leq \sqrt[4]{c} \end{cases}$$

The remaining singular values $\lambda_{r+1}(Y), \dots, \lambda_{m \wedge n}(Y)$ of Y are associated with the zero singular values of A : their empirical distribution converges weakly to the limiting distribution in Proposition 8.

Proposition 10. Let Y follow the asymptotic matrix reconstruction model (7) with distinct signal singular values $\lambda_1(A) > \lambda_2(A) > \dots > \lambda_r(A) > 0$. Fix j such that $\lambda_j(A) > \sqrt[4]{c}$. Then as n tends to infinity,

$$\langle u_j(Y), u_j(A) \rangle^2 \xrightarrow{P} \left(1 - \frac{c}{\lambda_j^4(A)}\right) / \left(1 + \frac{c}{\lambda_j^2(A)}\right)$$

and

$$\langle v_j(Y), v_j(A) \rangle^2 \xrightarrow{P} \left(1 - \frac{c}{\lambda_j^4(A)}\right) / \left(1 + \frac{1}{\lambda_j^2(A)}\right)$$

Moreover, if $k = 1, \dots, r$ not equal to j then $\langle u_j(Y), u_k(A) \rangle \xrightarrow{P} 0$ and $\langle v_j(Y), v_k(A) \rangle \xrightarrow{P} 0$ as n tends to infinity.

The limits established in Proposition 9 indicate a phase transition. If the singular value $\lambda_j(A)$ is less than or equal to $\sqrt[4]{c}$ then, asymptotically, the singular value $\lambda_j(Y)$ lies within the support of the Marčenko-Pastur distribution and is not distinguishable from the noise singular values. On the other hand, if $\lambda_j(A)$ exceeds $\sqrt[4]{c}$ then, asymptotically, $\lambda_j(Y)$ lies outside the support of the Marčenko-Pastur distribution, and the corresponding left and right singular vectors of Y are associated with those of A (Proposition 10).

4 Proposed Reconstruction Method

Assume for the moment that the variance σ^2 of the noise is known, and equal to one. Let Y be an observed $m \times n$ matrix generated from the additive model

$Y = A + n^{-1/2}W$, and let

$$Y = \sum_{j=1}^{m \wedge n} \lambda_j(Y) u_j(Y) v'_j(Y)$$

be the SVD of Y . Following the discussion in Section 2, we seek an estimate \hat{A} of the signal matrix A having the form

$$\hat{A} = \sum_{j=1}^{m \wedge n} c_j u_j(Y) v'_j(Y),$$

where each coefficient c_j depends only on the singular values $\lambda_1(Y), \dots, \lambda_{m \wedge n}(Y)$ of Y . We derive \hat{A} from the limiting relations in Propositions 9 and 10. By way of approximation, we treat these relations as exact in the non-asymptotic setting under study, using the symbols $\stackrel{l}{=}$, $\stackrel{l}{\leq}$ and $\stackrel{l}{>}$ to denote limiting equality and inequality relations.

Suppose initially that the singular values and vectors of the signal matrix A are known. In this case we wish to find coefficients $\{c_j\}$ minimizing

$$\text{Loss}(A, \hat{A}) = \left\| \sum_{j=1}^{m \wedge n} c_j u_j(Y) v'_j(Y) - \sum_{j=1}^r \lambda_j(A) u_j(A) v'_j(A) \right\|_F^2.$$

Proposition 9 shows that asymptotically the information about the singular values of A that are smaller than $\sqrt[4]{c}$ is not recoverable from the singular values of Y . Thus we can restrict the first sum to the first $r_0 = \#\{j : \lambda_j(A) > \sqrt[4]{c}\}$ terms

$$\text{Loss}(A, \hat{A}) = \left\| \sum_{j=1}^{r_0} c_j u_j(Y) v'_j(Y) - \sum_{j=1}^r \lambda_j(A) u_j(A) v'_j(A) \right\|_F^2.$$

Proposition 10 ensures that the left singular vectors $u_j(Y)$ and $u_k(A)$ are asymptotically orthogonal for $k = 1, \dots, r$ not equal to $j = 1, \dots, r_0$, and therefore

$$\text{Loss}(A, \hat{A}) \stackrel{l}{=} \sum_{j=1}^{r_0} \left\| c_j u_j(Y) v'_j(Y) - \lambda_j(A) u_j(A) v'_j(A) \right\|_F^2 + \sum_{j=r_0+1}^r \lambda_j^2(A).$$

Fix $1 \leq j \leq r_0$. Expanding the j -th term in the above sum gives

$$\begin{aligned}
& \|\lambda_j(A) u_j(A) v'_j(A) - c_j u_j(Y) v'_j(Y)\|_F^2 \\
&= c_j^2 \|u_j(Y) v'_j(Y)\|_F^2 + \lambda_j^2(A) \|u_j(A) v'_j(A)\|_F^2 \\
&\quad - 2c_j \lambda_j(A) \langle u_j(A) v'_j(A), u_j(Y) v'_j(Y) \rangle \\
&= \lambda_j^2(A) + c_j^2 - 2c_j \lambda_j(A) \langle u_j(A), u_j(Y) \rangle \langle v_j(A), v_j(Y) \rangle.
\end{aligned}$$

Differentiating the last expression with respect to c_j yields the optimal value

$$c_j^* = \lambda_j(A) \langle u_j(A), u_j(Y) \rangle \langle v_j(A), v_j(Y) \rangle. \quad (9)$$

In order to estimate the coefficient c_j^* we consider separately singular values of Y that are at most, or greater than $1 + \sqrt{c}$, where $c = m/n$ is the aspect ratio of Y . By Proposition 9, the asymptotic relation $\lambda_j(Y) \leq 1 + \sqrt{c}$ implies $\lambda_j(A) \leq \sqrt[4]{c}$, and in this case the j -th singular value of A is not recoverable from Y . Thus if $\lambda_j(Y) \leq 1 + \sqrt{c}$ we set the corresponding coefficient $c_j^* = 0$.

On the other hand, the asymptotic relation $\lambda_j(Y) > 1 + \sqrt{c}$ implies that $\lambda_j(A) > \sqrt[4]{c}$, and that each of the inner products in (9) are asymptotically positive. The displayed equations in Propositions 9 and 10 can then be used to obtain estimates of each term in (9) based only on the (observed) singular values of Y and its aspect ratio c . These equations yield the following relations:

$$\begin{aligned}
\widehat{\lambda}_j^2(A) &= \frac{1}{2} \left[\lambda_j^2(Y) - (1 + c) + \sqrt{[\lambda_j^2(Y) - (1 + c)]^2 - 4c} \right] \text{ estimates } \lambda_j^2(A), \\
\widehat{\theta}_j^2 &= \left(1 - \frac{c}{\widehat{\lambda}_j^4(A)} \right) / \left(1 + \frac{c}{\widehat{\lambda}_j^2(A)} \right) \text{ estimates } \langle u_j(A), u_j(Y) \rangle^2, \\
\widehat{\phi}_j^2 &= \left(1 - \frac{c}{\widehat{\lambda}_j^4(A)} \right) / \left(1 + \frac{1}{\widehat{\lambda}_j^2(A)} \right) \text{ estimates } \langle v_j(A), v_j(Y) \rangle^2.
\end{aligned}$$

With these estimates in hand, the proposed reconstruction scheme is defined via the equation

$$G_o^{RMT}(Y) = \sum_{\lambda_j(Y) > 1 + \sqrt{c}} \widehat{\lambda}_j(A) \widehat{\theta}_j \widehat{\phi}_j u_j(Y) v'_j(Y), \quad (10)$$

where $\hat{\lambda}_j(A)$, $\hat{\theta}_j$, and $\hat{\phi}_j$ are the positive square roots of the estimates defined above.

The RMT method shares features with both hard and soft thresholding. It sets to zero singular values of Y smaller than the threshold $(1 + \sqrt{c})$, and it shrinks the remaining singular values towards zero. However, unlike soft thresholding the amount of shrinkage depends on the singular values, the larger singular values are shrunk less than the smaller ones. This latter feature is similar to that of LASSO type estimators based on an L_q penalty with $0 < q < 1$ (also known as bridge estimators [Fu, 1998](#)). It is important to note that, unlike hard and soft thresholding schemes, the proposed RMT method has no tuning parameters. The only unknown, the noise variance, is estimated within the procedure.

In the general version of the matrix reconstruction problem, the variance σ^2 of the noise is not known. In this case, given an estimate $\hat{\sigma}^2$ of σ^2 , such as that described below, we may define

$$G^{RMT}(Y) = \hat{\sigma} G_o^{RMT} \left(\frac{Y}{\hat{\sigma}} \right), \quad (11)$$

where $G_o^{RMT}(\cdot)$ is the estimate defined in [\(10\)](#).

4.1 Estimation of the Noise Variance

Let Y be derived from the asymptotic reconstruction model $Y = A + \sigma n^{-1/2}W$ with sigma unknown. While it is natural to try to estimate σ from the entries of Y , the following general results indicate that, under mild conditions, it is sufficient to consider estimates based on the singular values of Y . The results and their proofs parallel those in Section [??](#).

Definition 11. *A function $s(\cdot) : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is orthogonally invariant if for any $m \times n$ matrix Y and any orthogonal matrices U and V of appropriate sizes, $s(Y) = s(UYV')$.*

Proposition 12. *A function $s(\cdot) : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is orthogonally invariant if and only if $s(Y)$ depends only on the singular values of Y .*

Proposition 13. *Let $s(\cdot) : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$. Then there is an orthogonally invariant function $\tilde{s}(\cdot)$ with the following property. Let \mathbf{A} and W be independent*

$m \times n$ random matrices with orthogonally invariant distributions, and let $Y = \mathbf{A} + \sigma n^{-1/2}W$ for some σ . Then $\tilde{s}(Y)$ has the same expected value as $s(Y)$ and a smaller or equal variance.

Based Propositions 12 and 13 we restrict our attention to the estimates of σ that depend only on the singular values of Y . It follows from Proposition 9 that the empirical distribution of the $(m-r)$ singular values $S = \{\lambda_j(Y/\sigma) : \lambda_j(A) = 0\}$ converges weakly to a distribution with density (8) supported on the interval $[|1 - \sqrt{c}|, 1 + \sqrt{c}]$. Following the general approach outlined in Györfi et al. (1996), we estimate σ by minimizing the Kolmogorov-Smirnov distance between the observed sample distribution of the singular values of Y and that predicted by theory. Let F be the CDF of the density (8). For each $\sigma > 0$ let \widehat{S}_σ be the set of singular values $\lambda_j(Y)$ that fall in the interval $[\sigma|1 - \sqrt{c}|, \sigma(1 + \sqrt{c})]$, and let \widehat{F}_σ be the empirical CDF of \widehat{S}_σ . Then

$$K(\sigma) = \sup_s |F(s/\sigma) - \widehat{F}_\sigma(s)|$$

is the Kolmogorov-Smirnov distance between the empirical and theoretical singular value distribution functions, and we define

$$\hat{\sigma}(Y) = \arg \min_{\sigma > 0} K(\sigma) \quad (12)$$

to be the value of σ minimizing $K(\sigma)$. A routine argument shows that the estimator $\hat{\sigma}$ is scale invariant, in the sense that $\hat{\sigma}(\beta Y) = \beta \hat{\sigma}(Y)$ for each $\beta > 0$.

By considering the jump points of the empirical CDF $\widehat{F}_\sigma(s)$, the supremum in $K(\sigma)$ simplifies to

$$K(\sigma) = \max_{s_i \in \widehat{S}_\sigma} \left| F(s_i/\sigma) - \frac{i - 1/2}{|\widehat{S}_\sigma|} \right| + \frac{1}{2|\widehat{S}_\sigma|},$$

where $\{s_i\}$ are the ordered elements of \widehat{S}_σ . The objective function $K(\sigma)$ is discontinuous at points where the \widehat{S}_σ changes, so we minimize it over a fine grid of points σ in the range where $|\widehat{S}_\sigma| > (m \wedge n)/2$ and $\sigma(1 + \sqrt{c}) < 2\lambda_1(Y)$. The closed form of the cumulative distribution function $F(\cdot)$ is presented in Section 5.4.

5 Simulations

We carried out a simulation study to evaluate the performance of the RMT reconstruction scheme $G^{RMT}(\cdot)$ defined in (11) using the variance estimate $\hat{\sigma}$ in (12). The study compared the performance of $G^{RMT}(\cdot)$ to three alternatives: the best hard thresholding reconstruction scheme, the best soft thresholding reconstruction scheme, and the best orthogonally invariant reconstruction scheme. Each of the three competing alternatives is an oracle-type procedure that is based on information about the signal matrix A that is not available to $G^{RMT}(\cdot)$.

5.1 Hard and Soft Thresholding Oracle Procedures

Hard and soft thresholding schemes require specification of a threshold parameter that can depend on the observed matrix Y . Estimation of the noise variance can be incorporated into the choice of the threshold parameter. In order to compare the performance of $G^{RMT}(\cdot)$ against every possible hard and soft thresholding scheme, we define oracle procedures

$$G^H(Y) = g_{\lambda^*}^H(Y) \text{ where } \lambda^* = \arg \min_{\lambda > 0} \|A - g_\lambda^H(Y)\|_F^2 \quad (13)$$

$$G^S(Y) = g_{\nu^*}^S(Y) \text{ where } \nu^* = \arg \min_{\nu > 0} \|A - g_\nu^S(Y)\|_F^2 \quad (14)$$

that make use of the signal A . By definition, the loss $\|A - G^H(Y)\|_F^2$ of $G^H(Y)$ is less than that of any hard thresholding scheme, and similarly the loss of $G^S(Y)$ is less than that of any soft thresholding procedure. In effect, the oracle procedures have access to both the unknown signal matrix A and the unknown variance σ . They are constrained only by the form of their respective thresholding families. The oracle procedures are not realizable in practice.

5.2 Orthogonally Invariant Oracle Procedure

As shown in Corrolary 7, every orthogonally invariant reconstruction scheme $g(\cdot)$ has the form

$$g(Y) = \sum_{j=1}^{m \wedge n} c_j u_j(Y) v_j(Y)',$$

where the coefficients c_j are functions of the singular values of Y .

The orthogonally invariant oracle scheme has coefficients c_j^o minimizing the loss

$$\left\| A - \sum_{j=1}^{m \wedge n} c_j u_j(Y) v_j(Y)' \right\|_F^2$$

over all choices c_j . As in the case with the hard and soft thresholding oracle schemes, the coefficients c_j^o depend on the signal matrix A , which in practice is unknown.

The (rank one) matrices $\{u_j(Y)v_j(Y)'\}$ form an orthonormal basis of an $m \wedge n$ -dimensional subspace of the mn -dimensional space of all $m \times n$ matrices. Thus the optimal coefficient c_j^o is simply the matrix inner product $\langle A, u_j(Y)v_j(Y)' \rangle$, and the orthogonally invariant oracle scheme has the form of a projection

$$G^*(Y) = \sum_{j=1}^{m \wedge n} \langle A, u_j(Y)v_j(Y)' \rangle u_j(Y)v_j(Y)'. \quad (15)$$

By definition, for any orthogonally invariant reconstruction scheme $g(\cdot)$ and observed matrix Y , we have $\|A - G^*(Y)\|_F^2 \leq \|A - g(Y)\|_F^2$.

5.3 Simulations

We compared the reconstruction schemes $G^H(Y)$, $G^S(Y)$, and $G^{RMT}(Y)$ to $G^*(Y)$ on a wide variety of signal matrices generated according to the model (1). As shown in Proposition 5, the distribution of the loss $\|A - G(Y)\|_F^2$ depends only on the singular values of A , so we considered only diagonal signal matrices. As the variance estimate used in $G^{RMT}(\cdot)$ is scale invariant, all simulations were run with noise of unit variance. (Estimation of noise variance is not necessary for the oracle reconstruction schemes.)

5.3.1 Square Matrices

Our initial simulations considered 1000×1000 square matrices. Signal matrices A were generated using three parameters: the rank r ; the largest singular value $\lambda_1(A)$; and the decay profile of the remaining singular values. We considered ranks $r \in \{1, 3, 10, 32, 100\}$ corresponding to successive powers of $\sqrt{10}$ up to $(m \wedge n)/10$, and maximum singular values $\lambda_1(A) \in \{0.9, 1, 1.1, \dots, 10\} \sqrt[4]{c}$

falling below and above the critical threshold of $\sqrt[4]{c} = 1$. We considered several coefficient decay profiles: (i) all coefficients equal; (ii) linear decay to zero; (iii) linear decay to $\lambda_1(A)/2$; and (iv) exponential decay as powers of 0.5, 0.7, 0.9, 0.95, or 0.99. Independent noise matrices W were generated for each signal matrix A . All reconstruction schemes were then applied to the resulting matrix $Y = A + n^{-1/2}W$. The total number of generated signal matrices was 3,680.

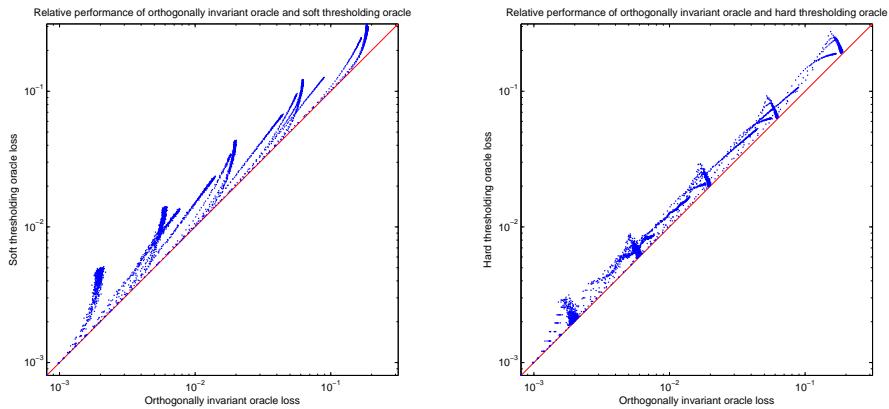


Figure 3: Relative performance of soft and hard thresholding method against the orthogonally invariant oracle for 1000×1000 matrices.

Figures 3 and 4 illustrate, respectively, the loss of the best soft thresholding, best hard thresholding and RMT reconstruction methods (y axis) relative to the best orthogonally invariant scheme (x axis). In each case the diagonal represents the performance of the orthogonally invariant oracle: points farther from the diagonal represent worse performance. The plots show clearly that $G^{RMT}(\cdot)$ outperforms the oracle schemes $G^H(\cdot)$ and $G^S(\cdot)$, and has performance comparable to that of the orthogonally invariant oracle. In particular, $G^{RMT}(\cdot)$ outperforms any hard or soft thresholding scheme, even if the latter schemes have access to the unknown variance σ and the signal matrix A .

In order to summarize the results of our simulations, for each scheme $G(\cdot)$ and for each matrix Y generated from a signal matrix A we calculated the

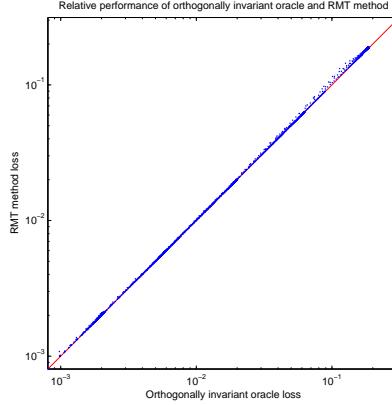


Figure 4: Relative performance of RMT method and orthogonally invariant oracle method for 1000×1000 matrices.

relative excess loss of $G()$ with respect to $G^*(())$:

$$\text{REL}(A, G(Y)) = \frac{\text{Loss}(A, G(Y))}{\text{Loss}(A, G^*(Y))} - 1 \quad (16)$$

The definition of $G^*(())$ ensures that relative excess loss is non-negative. The average REL of $G^S(\cdot)$, $G^H(\cdot)$, and $G^{RMT}(\cdot)$ across the 3680 simulated 1000×1000 matrices was 68.3%, 18.3%, and 0.61% respectively. Table 1 summarizes these results, and the results of analogous simulations carried out on square matrices of different dimensions. The table clearly shows the strong performance of RMT method for matrices with at least 50 rows and columns. Even for $m = n = 50$, the average relative excess loss of the RMT method is almost twice smaller than those of the oracle soft and hard thresholding methods.

5.3.2 Rectangular Matrices

We performed simulations for rectangular matrices of different dimensions m, n and different aspect ratios $c = m/n$. For each choice of dimensions m, n we simulated target matrices using the same rules as in the square case: rank $r \in \{1, 3, 10, 32, \dots\}$ not exceeding $(m \wedge n)/10$, maximum singular values

Table 1: Average relative excess losses of oracle soft thresholding, oracle hard thresholding and the proposed RMT reconstruction method for square matrices of different dimensions.

Matrix size (square)		2000	1000	500	100	50
Scheme	$G^S(\cdot)$	0.740	0.683	0.694	0.611	0.640
	$G^H(\cdot)$	0.182	0.183	0.178	0.179	0.176
	$G^{RMT}(\cdot)$	0.003	0.006	0.008	0.029	0.071

$\lambda_1(A) \in \{0.9, 1, 1.1, \dots, 10\} \sqrt[4]{c}$, and coefficients decay profiles like those above. A summary of the results is given in Table 2, which shows the average REL for matrices with 2000 rows and 10 to 2000 columns. Although random matrix theory used to construct the RMT scheme requires m and n to tend to infinity and at the same rate, the numbers in Table 2 clearly show that the performance of the RMT scheme is excellent even for small n , where average REL ranges between 0.3% and 0.54%. The average REL of soft and hard thresholding are above 18% in each of the simulations.

Table 2: Average relative excess loss of oracle soft thresholding, oracle hard thresholding, and RMT reconstruction schemes for matrices with different dimensions and aspect ratios.

Matrix size	m	2000	2000	2000	2000	2000	2000
	n	2000	1000	500	100	50	10
Method	$G^S(\cdot)$	0.740	0.686	0.653	0.442	0.391	0.243
	$G^H(\cdot)$	0.182	0.188	0.198	0.263	0.292	0.379
	$G^{RMT}(\cdot)$	0.003	0.004	0.004	0.004	0.004	0.005

Acknowledgments

This work was supported, in part, by grants from US EPA (RD832720 and RD833825) and grant from NSF (DMS-0907177).

The authors would like to thank Florentina Bunea and Marten Wegkamp for helpful discussions.

Appendix

5.4 Cumulative Distribution Function for Variance Estimation

The cumulative density function $F(\cdot)$ is calculated as the integral of $f_{n^{-1/2}W}(s)$. For $c = 1$ ($a = 0, b = 4$) it is a common integral

$$F(x) = \int_{\sqrt{a}}^x f(s)ds = \frac{1}{\pi} \int_0^x \sqrt{b-s^2}ds = \frac{1}{2\pi} \left(x\sqrt{4-x^2} + 4 \arcsin \frac{x}{2} \right)$$

For $c \neq 1$ the calculations are more complicated. First we perform the change of variables $t = s^2$, which yields

$$\begin{aligned} F(x) &= \int_{\sqrt{a}}^x f(s)ds = C \int_{\sqrt{a}}^x s^{-2} \sqrt{(b-s^2)(s^2-a)}ds^2 \\ &= C \int_a^{x^2} t^{-1} \sqrt{(b-t)(t-a)}dt, \end{aligned}$$

where $C = 1/(2\pi(c \wedge 1))$.

Next we perform a change of variables $y = t - [a+b]/2$ to make the expression in the square root look like $h^2 - x^2$, giving

$$\begin{aligned} F(x) &= C \int_{-[b-a]/2}^{x^2-[a+b]/2} \frac{\sqrt{([b-a]/2-y)(y+[b-a]/2)}}{y+[a+b]/2} dy \\ &= C \int_{-2\sqrt{c}}^{x^2-(1+c)} \frac{\sqrt{4c-y^2}}{y+1+c} dy, \end{aligned}$$

The second equality above uses the fact that $a+b = 2(1+c)$ and $b-a = 4\sqrt{c}$. The simple change of variables $y = 2\sqrt{c}z$ is performed next to make the numerator $\sqrt{1-z^2}$:

$$F(x) = \frac{\sqrt{c}}{\pi(c \wedge 1)} \int_{-1}^{[x^2-(1+c)]/2\sqrt{c}} \frac{\sqrt{1-z^2}}{z+(1+c)/2\sqrt{c}} dz$$

Next, the formula

$$\begin{aligned} \int \frac{\sqrt{1-z^2}}{z+q} dz &= \sqrt{1-z^2} + q \arcsin(z) \\ &\quad - \sqrt{q^2-1} \arctan \left[\frac{qz+1}{\sqrt{(q^2-1)(1-z^2)}} \right] \end{aligned}$$

is applied to find the closed form of $F(x)$ by substituting $z = [x^2 - (1 + c)]/2\sqrt{c}$ and $q = (1 + c)/2\sqrt{c}$. The final expression above can be simplified as $\sqrt{q^2 - 1} = \sqrt{[(1 + c)/2\sqrt{c}]^2 - 1} = |1 - c|/2\sqrt{c}$.

5.5 Limit Theorems for Asymptotic Matrix Reconstruction Problem

Propositions 9 and 10 in Section 3 provide an asymptotic connection between the eigenvalues and eigenvectors of the signal matrix A and those of the observed matrix Y . Each proposition is derived from recent work in random matrix theory on spiked population models. Spiked population models were introduced by [Johnstone \(2001\)](#).

5.5.1 The Spiked Population Model

The spiked population model is formally defined as follows. Let $r \geq 1$ and constants $\tau_1 \geq \dots \geq \tau_r > 1$ be given, and for $n \geq 1$ let integers $m = m(n)$ be defined in such a way that

$$\frac{m}{n} \rightarrow c > 0 \text{ as } n \rightarrow \infty. \quad (17)$$

For each n let

$$T = \text{diag}(\tau_1, \dots, \tau_r, 1, \dots, 1)$$

be an $m \times m$ diagonal matrix (with $m = m(n)$), and let X be an $m \times n$ matrix with independent $N_m(0, T)$ columns. Let $\widehat{T} = n^{-1}XX'$ be the sample covariance matrix of X .

The matrix X appearing in the spiked population model may be decomposed as a sum of matrices that parallel those in the matrix reconstruction problem. In particular, X can be represented as a sum

$$X = X_1 + Z, \quad (18)$$

where X_1 has independent $N_m(0, T - I)$ columns, Z has independent $N(0, 1)$ entries, and X_1 and Z are independent. It follows from the definition of T that

$$(T - I) = \text{diag}(\tau_1 - 1, \dots, \tau_r - 1, 0, \dots, 0),$$

and therefore the entries in rows $r+1, \dots, m$ of X_1 are equal to zero. Thus, the sample covariance matrix $\widehat{T}_1 = n^{-1}X_1X_1'$ of X_1 has the simple block form

$$\widehat{T}_1 = \left[\begin{array}{c|c} \widehat{T}_{11} & 0 \\ \hline 0 & 0 \end{array} \right]$$

where \widehat{T}_{11} is an $r \times r$ matrix equal to the sample covariance of the first r rows of X_1 . It is clear from the block structure that the first r eigenvalues of \widehat{T}_1 are equal to the eigenvalues of \widehat{T}_{11} , and that the remaining $(m-r)$ eigenvalues of \widehat{T}_1 are equal to zero. The size of \widehat{T}_{11} is fixed, and therefore as n tends to infinity, its entries converge in probability to those of $\text{diag}(\tau_1 - 1, \dots, \tau_r - 1)$. In particular,

$$\left\| \frac{1}{n}X_1X_1' - (T - I) \right\|_F^2 \xrightarrow{P} 0. \quad (19)$$

Consequently, for each $j = 1, \dots, r$, as n tends to infinity

$$\lambda_j^2(n^{-1/2}X_1) = \lambda_j(\widehat{T}_1) = \lambda_j(\widehat{T}_{11}) \xrightarrow{P} \tau_j - 1 \quad (20)$$

and

$$\langle u_j(\widehat{T}_{11}), e_j \rangle^2 \xrightarrow{P} 1, \quad (21)$$

where e_j is the j -th canonical basis element in \mathbb{R}^r . An easy argument shows that $u_j(n^{-1/2}X_1) = u_j(\widehat{T}_1)$, and it then follows from (21) that

$$\langle u_j(n^{-1/2}X_1), e_j \rangle^2 \xrightarrow{P} 1, \quad (22)$$

where e_j is the j -th canonical basis element in \mathbb{R}^m .

5.5.2 Proof of Proposition 9

Proposition 9 is derived from existing results on the limiting singular values of \widehat{T} in the spiked population model. These results are summarized in the following theorem, which is a combination of Theorems 1.1, 1.2 and 1.3 in [Baik and Silverstein \(2006\)](#).

Theorem A. *If \widehat{T} is derived from the spiked population model with parameters $\tau_1, \dots, \tau_r > 1$, then for $j = 1, \dots, r$, as $n \rightarrow \infty$*

$$\lambda_j(\widehat{T}) \xrightarrow{P} \begin{cases} \tau_j + c \frac{\tau_j}{\tau_j - 1} & \text{if } \tau_j > 1 + \sqrt{c} \\ (1 + \sqrt{c})^2 & \text{if } 1 < \tau_j \leq 1 + \sqrt{c} \end{cases}$$

The remaining sample eigenvalues $\lambda_{r+1}(\hat{T}), \dots, \lambda_{m \wedge n}(\hat{T})$ are associated with the unit eigenvalues of T , their empirical distribution converges weakly to the Marčenko-Pastur distribution.

We also require the following inequality of [Mirsky \(1960\)](#).

Theorem B. *If B and C are $m \times n$ matrices then $\sum_{j=1}^{m \wedge n} [\lambda_j(C) - \lambda_j(B)]^2 \leq \|C - B\|_F^2$.*

Proof of Proposition 9. Fix $n \geq 1$ and let Y follow the asymptotic reconstruction model (7), where the signal matrix A has fixed rank r and non-zero singular values $\lambda_1(A), \dots, \lambda_r(A)$. Based on orthogonal invariance of the matrix reconstruction problem, without loss of generality, we will assume that the signal matrix $A = \text{diag}(\lambda_1(A), \dots, \lambda_r(A), 0, \dots, 0)$.

We begin by considering a spiked population model whose parameters match those of the matrix reconstruction model. Let X have the same dimensions as Y and be derived from a spiked population model with covariance matrix T having r non-unit eigenvalues

$$\tau_j = \lambda_j^2(A) + 1, \quad j = 1, \dots, r. \quad (23)$$

As noted above, we may represent X as $X = X_1 + Z$, where X_1 has independent $N(0, T - I)$ columns, Z has independent $N(0, 1)$ entries and X_1 is independent of Z . Recall that the limiting relations (19)-(22) hold for this representation.

The matrix reconstruction problem and spiked population model may be coupled in a natural way. Let random orthogonal matrices U_1 and V_1 be defined for each sample point in such a way that $U_1 D_1 V_1'$ is the SVD of X_1 . By construction, the matrices U_1, V_1 depend only on X_1 , and are therefore independent of Z . Consequently $U_1' Z V_1$ has the same distribution as Z . If we define $\tilde{W} = U_1' Z V_1$, then $\tilde{Y} = A + n^{-1/2} \tilde{W}$ has the same distribution as the observed matrix Y in the matrix reconstruction problem.

We apply Mirsky's theorem with $B = \tilde{Y}$ and $C = n^{-1/2} U_1' X V_1$ in order to

bound the difference between the singular values of \tilde{Y} and those of $n^{-1/2}X$:

$$\begin{aligned}
\sum_{j=1}^{m \wedge n} [\lambda_j(n^{-1/2}X) - \lambda_j(\tilde{Y})]^2 &\leq \|n^{-1/2}U_1'XV_1 - \tilde{Y}\|_F^2 \\
&= \|(n^{-1/2}U_1'X_1V_1 - A) + n^{-1/2}(U_1'ZV_1 - \tilde{W})\|_F^2 \\
&= \|n^{-1/2}U_1'X_1V_1 - A\|_F^2 \\
&= \sum_{j=1}^{m \wedge n} [\lambda_j(n^{-1/2}U_1'X_1V_1) - \lambda_j(A)]^2 \\
&= \sum_{j=1}^{m \wedge n} [\lambda_j(n^{-1/2}X_1) - \lambda_j(A)]^2.
\end{aligned}$$

The first inequality above follows from Mirsky's theorem and the fact that the singular values of $n^{-1/2}X$ and $n^{-1/2}U_1'XV_1$ are the same, even though U_1 and V_1 may not be independent of X . The next two equalities follow by expanding X and \tilde{Y} , and the fact that $\tilde{W} = U_1'ZV_1$. The third equality is a consequence of the fact that both $U_1'X_1V_1$ and A are diagonal, and the final equality follows from the equality of the singular values of X_1 and $U_1'X_1V_1$. In conjunction with (20) and (23), the last display implies that

$$\sum_j [\lambda_j(n^{-1/2}X) - \lambda_j(\tilde{Y})]^2 \xrightarrow{P} 0.$$

Thus the distributional and limit results for the eigenvalues of $\hat{T} = n^{-1}XX'$ hold also for the eigenvalues of $\tilde{Y}\tilde{Y}'$, and therefore for YY' as well. The relation $\lambda_j(Y) = \sqrt{\lambda_j(YY')}$ completes the proof. \square

5.5.3 Proof of Proposition 10

Proposition 10 may be derived from existing results on the limiting singular vectors of the sample covariance \hat{T} in the spiked population model. These results are summarized in Theorem C below. The result was first established for Gaussian models and aspect ratios $0 < c < 1$ by [Paul \(2007\)](#). [Nadler \(2008\)](#) extended Paul's results to $c > 0$. Recently [Lee et al. \(2010\)](#) further extended the theorem to $c \geq 0$ and non-Gaussian models.

Theorem C. *If \widehat{T} is derived from the spiked population model with distinct parameters $\tau_1 > \dots > \tau_r > 1$, then for $1 \leq j \leq r$,*

$$\langle u_j(\widehat{T}), u_j(T) \rangle^2 \xrightarrow{P} \begin{cases} \left(1 - \frac{c}{(\tau_j-1)^2}\right) / \left(1 + \frac{c}{\tau_j-1}\right) & \text{if } \tau_j > 1 + \sqrt{c} \\ 0 & \text{if } 1 < \tau_j \leq 1 + \sqrt{c} \end{cases}$$

Moreover, for $\tau_j > 1 + \sqrt{c}$ and $k \neq j$ such that $1 \leq k \leq r$ we have

$$\langle u_j(\widehat{T}), u_k(T) \rangle^2 \xrightarrow{P} 0.$$

Although the last result is not explicitly stated in [Paul \(2007\)](#), it follows immediately from the central limit theorem for eigenvectors (Theorem 5, [Paul, 2007](#)).

We also require the following result, which is a special case of an inequality of Wedin ([Wedin, 1972](#); [Stewart, 1991](#)).

Theorem D. *Let B and C be $m \times n$ matrices and let $1 \leq j \leq m \wedge n$. If the j -th singular value of C is separated from the singular values of B and bounded away from zero, in the sense that for some $\delta > 0$*

$$\min_{k \neq j} |\lambda_j(C) - \lambda_k(B)| > \delta \quad \text{and} \quad \lambda_j(C) > \delta$$

then

$$\langle u_j(B), u_j(C) \rangle^2 + \langle v_j(B), v_j(C) \rangle^2 \geq 2 - \frac{2\|B - C\|_F^2}{\delta^2}.$$

Proof of Proposition 10: Fix $n \geq 1$ and let Y follow the asymptotic reconstruction model (7), where the signal matrix A has fixed rank r and non-zero singular values $\lambda_1(A), \dots, \lambda_r(A)$. Assume without loss of generality that $A = \text{diag}(\lambda_1(A), \dots, \lambda_r(A), 0, \dots, 0)$.

We consider a spiked population model whose parameters match those of the matrix reconstruction problem and couple it with the matrix reconstruction model exactly as in the proof of Proposition 9. In particular, the quantities $\tau_j, T, X, X_1, Z, U_1, V_1, \tilde{W}$, and \tilde{Y} are as in the proof of Proposition 9 and the preceding discussion.

Fix an index j such that $\lambda_j(A) > \sqrt[4]{c}$ and thus $\tau_j > 1 + \sqrt{c}$. We apply Wedin's theorem with $B = \tilde{Y}$ and $C = n^{-1/2}U_1'XV_1$. There is $\delta > 0$ such that both conditions of Wedin's theorem are satisfied for the given j with probability converging to 1 as $n \rightarrow \infty$. The precise choice of δ is presented

at the end of this proof. It follows from Wedin's theorem and inequality $\langle v_j(B), v_j(C) \rangle^2 \leq 1$ that

$$\langle u_j(\tilde{Y}), u_j(n^{-1/2}U_1'XV_1) \rangle^2 = \langle u_j(B), u_j(C) \rangle^2 \geq 1 - \frac{2\|B - C\|_F^2}{\delta^2}$$

It is shown in the proof of Proposition 9 that $\|B - C\|_F^2 = \|n^{-1/2}U_1'XV_1 - \tilde{Y}\|_F^2 \xrightarrow{P} 0$ as $n \rightarrow \infty$. Substituting $u_j(n^{-1/2}U_1'XV) = U_1'u_j(X)$ then yields

$$\langle u_j(\tilde{Y}), U_1'u_j(X) \rangle^2 \xrightarrow{P} 1. \quad (24)$$

Fix $k = 1, \dots, r$. As $\tau_j > 1 + \sqrt{c}$ Theorem C shows that $\langle u_j(\tilde{T}), e_k \rangle^2$ has a non-random limit in probability, which we will denote by θ_{jk}^2 . The relation $\tau_j = \lambda_j^2(A) + 1$ implies that $\theta_{jk}^2 = [1 - c\lambda_j^{-4}(A)]/[1 + c\lambda_j^{-2}(A)]$ if $j = k$, and $\theta_{jk}^2 = 0$ otherwise. As $u_j(\tilde{T}) = u_j(X)$, it follows that

$$\langle u_j(X), e_k \rangle^2 \xrightarrow{P} \theta_{jk}^2.$$

Recall that the matrix U_1 consists of the left singular vectors of X_1 , i.e. $u_k(X_1) = U_1e_k$. It is shown in (22) that $\langle U_1e_k, e_k \rangle^2 = \langle u_k(X_1), e_k \rangle^2 \xrightarrow{P} 1$, so we can replace e_k by U_1e_k in the previous display to obtain

$$\langle u_j(X), U_1e_k \rangle^2 \xrightarrow{P} \theta_{jk}^2.$$

It follows from the basic properties of inner products that

$$\langle U_1'u_j(X), e_k \rangle^2 \xrightarrow{P} \theta_{jk}^2.$$

Using the result (24) of Wedin's theorem we may replace the left term in the inner product by $u_j(\tilde{Y})$, which yields

$$\langle u_j(\tilde{Y}), e_k \rangle^2 \xrightarrow{P} \theta_{jk}^2.$$

As $A = \text{diag}(\lambda_1(A), \dots, \lambda_r(A), 0, \dots, 0)$ we have $e_k = u_k(A)$. By construction the matrix \tilde{Y} has the same distribution as Y , so it follows from the last display that

$$\langle u_j(Y), u_k(A) \rangle^2 \xrightarrow{P} \theta_{jk}^2,$$

which is equivalent to the statement of Proposition 10 for the left singular vectors. The statement for the right singular vectors follows from consideration of the transposed reconstruction problem.

Now we find such $\delta > 0$ that for the fixed j the conditions of Wedin's theorem are satisfied with probability going to 1. It follows from Proposition 9 that for $k = 1, \dots, r$ the k -th singular value of Y has a non-random limit in probability

$$\lambda_k^* = \lim \lambda_k(n^{-1/2}X) = \lim \lambda_k(\tilde{Y}).$$

Let r_0 be the number of eigenvalues of A such that $\lambda_k(A) > \sqrt[4]{c}$ (i.e. the inequality holds only for $k = 1, \dots, r_0$). It follows from the formula for λ_k^* that $\lambda_k^* > 1 + \sqrt{c}$ for $k = 1, \dots, r_0$. Note also that in this case λ_k^* is a strictly increasing function of $\lambda_k(A)$. All non-zero $\lambda_j(A)$ are distinct by assumption, so all λ_k^* are distinct for $k = 1, \dots, r_0$. Note that $\lambda_{r_0+1}^* = 1 + \sqrt{c}$ is smaller than $\lambda_{r_0}^*$. Thus the limits of the first r_0 singular values of Y are not only distinct, they are bounded away from all other singular values. Define

$$\delta = \frac{1}{3} \min_{k=1, \dots, r_0} (\lambda_k^* - \lambda_{k+1}^*) > 0.$$

For any $k = 1, \dots, r_0 + 1$ the following inequalities are satisfied with probability going to 1 as $n \rightarrow \infty$

$$|\lambda_k(Y) - \lambda_k^*| < \delta \quad \text{and} \quad |\lambda_k(n^{-1/2}X) - \lambda_k^*| < \delta. \quad (25)$$

In applying Wedin's theorem to $B = \tilde{Y}$ and $C = n^{-1/2}U_1'XV_1$ we must verify that for any $j = 1, \dots, r_0$ its two conditions are satisfied with probability going to 1. The first condition is $\lambda_j(C) > \delta$. When inequalities (25) hold

$$\begin{aligned} \lambda_j(C) &= \lambda_j(n^{-1/2}U_1'XV_1) = \lambda_j(n^{-1/2}X) > \lambda_j^* - \delta \\ &> (\lambda_j^* - \lambda_{j+1}^*) - \delta > 3\delta - \delta = 2\delta, \end{aligned} \quad (26)$$

so the first condition is satisfied with probability going to 1. The second condition is $|\lambda_j(C) - \lambda_k(B)| > \delta$ for all $k \neq j$. It is sufficient to check the condition for $k = 1, \dots, r_0 + 1$ as asymptotically $\lambda_j(C) > \lambda_{r_0+1}(B)$. From the definition of δ and the triangle inequality we get

$$3\delta < |\lambda_j^* - \lambda_k^*| \leq |\lambda_j^* - \lambda_j(n^{-1/2}X)| + |\lambda_j(n^{-1/2}X) - \lambda_k(\tilde{Y})| + |\lambda_k(\tilde{Y}) - \lambda_k^*|.$$

When inequalities (25) hold the first and the last terms on the right hand side sum are no larger than δ , thus

$$3\delta < \delta + |\lambda_j(n^{-1/2}X) - \lambda_k(\tilde{Y})| + \delta.$$

It follows that the second condition $|\lambda_j(C) - \lambda_k(B)| = |\lambda_j(n^{-1/2}X) - \lambda_k(\tilde{Y})| > \delta$ also holds with probability going to 1. \square

References

ALTER, O., BROWN, P., AND BOTSTEIN, D. 2000. Singular value decomposition for genome-wide expression data processing and modeling. *Proceedings of the National Academy of Sciences* 97, 18, 10101.

BAIK, J. AND SILVERSTEIN, J. 2006. Eigenvalues of large sample covariance matrices of spiked population models. *Journal of Multivariate Analysis* 97, 6, 1382–1408.

BUNEA, F., SHE, Y., AND WEGKAMP, M. 2010. Adaptive Rank Penalized Estimators in Multivariate Regression. *Arxiv preprint arXiv:1004.2995*.

CANDES, E. AND RECHT, B. Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, 1–56.

CANDÈS, E., ROMBERG, J., AND TAO, T. 2006. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Transactions on information theory* 52, 2, 489–509.

CAPITAIN, M., DONATI-MARTIN, C., AND FÉRAL, D. 2009. The largest eigenvalue of finite rank deformation of large Wigner matrices: convergence and non-universality of the fluctuations. *The Annals of Probability* 37, 1, 1–47.

DONOHO, D. 2006. Compressed sensing. *IEEE Transactions on Information Theory* 52, 4, 1289–1306.

DOZIER, R. AND SILVERSTEIN, J. 2007. On the empirical distribution of eigenvalues of large dimensional information-plus-noise-type matrices. *Journal of Multivariate Analysis* 98, 4, 678–694.

EL KAROUI, N. 2008. Spectrum estimation for large dimensional covariance matrices using random matrix theory. *The Annals of Statistics* 36, 6, 2757–2790.

FÉRAL, D. AND PÉCHÉ, S. 2007. The largest eigenvalue of rank one deformation of large Wigner matrices. *Communications in Mathematical Physics* 272, 1, 185–228.

FU, W. 1998. Penalized regressions: the bridge versus the lasso. *Journal of computational and graphical statistics* 7, 3, 397–416.

GEMAN, S. 1980. A limit theorem for the norm of random matrices. *The Annals of Probability* 8, 2, 252–261.

GYÖRFI, L., VAJDA, I., AND VAN DER MEULEN, E. 1996. Minimum Kolmogorov distance estimates of parameters and parametrized distributions. *Metrika* 43, 1, 237–255.

HOFMANN, K. AND MORRIS, S. 2006. *The structure of compact groups: a primer for the student, a handbook for the expert*. Walter De Gruyter Inc.

HOLTER, N., MITRA, M., MARITAN, A., CIEPLAK, M., BANAVAR, J., AND FEDOROFF, N. 2000. Fundamental patterns underlying gene expression profiles: simplicity from complexity. *Proceedings of the National Academy of Sciences* 97, 15, 8409.

JOHNSTONE, I. 2001. On the distribution of the largest eigenvalue in principal components analysis. *The Annals of Statistics* 29, 2, 295–327.

KONSTANTINIDES, K., NATARAJAN, B., AND YOVANOF, G. 1997. Noise estimation and filtering using block-based singular value decomposition. *IEEE Transactions on Image Processing* 6, 3, 479–483.

LEE, S., ZOU, F., AND WRIGHT, F. A. 2010. Convergence and Prediction of Principal Component Scores in High-Dimensional Settings. *The Annals of Statistics*.

MAIDA, M. 2007. Large deviations for the largest eigenvalue of rank one deformations of Gaussian ensembles. *The Electronic Journal of Probability* 12, 1131–1150.

MARČENKO, V. AND PASTUR, L. 1967. Distribution of eigenvalues for some sets of random matrices. *USSR Sbornik: Mathematics* 1, 4, 457–483.

MIRSKY, L. 1960. Symmetric gauge functions and unitarily invariant norms. *The Quarterly Journal of Mathematics* 11, 1, 50.

NADAKUDITI, R. AND SILVERSTEIN, J. 2007. Fundamental limit of sample eigenvalue based detection of signals in colored noise using relatively few samples. *Signals, Systems and Computers*, 686–690.

NADLER, B. 2008. Finite sample approximation results for principal component analysis: A matrix perturbation approach. *The Annals of Statistics* 36, 6, 2791–2817.

NEGAHBAN, S. AND WAINWRIGHT, M. 2009. Estimation of (near) low-rank matrices with noise and high-dimensional scaling. *Arxiv preprint arXiv:0912.5100*.

PAUL, D. 2007. Asymptotics of sample eigenstructure for a large dimensional spiked covariance model. *Statistica Sinica* 17, 4, 1617.

PÉCHÉ, S. 2006. The largest eigenvalue of small rank perturbations of Hermitian random matrices. *Probability Theory and Related Fields* 134, 1, 127–173.

RAYCHAUDHURI, S., STUART, J., AND ALTMAN, R. 2000. Principal components analysis to summarize microarray experiments: Application to sporulation time series. In *in Pacific Symposium on Biocomputing*. 452–463.

STEWART, G. 1991. Perturbation theory for the singular value decomposition. *SVD and Signal Processing, II: Algorithms, Analysis and Applications*, 99–109.

TROYANSKAYA, O., CANTOR, M., SHERLOCK, G., BROWN, P., HASTIE, T., TIBSHIRANI, R., BOTSTEIN, D., AND ALTMAN, R. 2001. Missing value estimation methods for DNA microarrays. *Bioinformatics* 17, 6, 520.

WACHTER, K. 1978. The strong limits of random matrix spectra for sample matrices of independent elements. *The Annals of Probability* 6, 1, 1–18.

WALL, M., DYCK, P., AND BRETTIN, T. 2001. SVDMAN–singular value decomposition analysis of microarray data. *Bioinformatics* 17, 6, 566.

WEDIN, P. 1972. Perturbation bounds in connection with singular value decomposition. *BIT Numerical Mathematics* 12, 1, 99–111.

WONGSAWAT, Y., RAO, K., AND ORAINTARA, S. Multichannel SVD-based image de-noising.